Ordering dynamics of the driven lattice-gas model

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The evolution of a two-dimensional driven lattice-gas model is studied on an $L_x \times L_y$ lattice. Scaling arguments and extensive numerical simulations are used to show that starting from random initial configuration the model evolves via two stages: (a) an early stage in which alternating stripes of particles and vacancies are formed along the direction y of the driving field, and (b) a stripe coarsening stage, in which the number of stripes is reduced and their average width increases. The number of stripes formed at the end of the first stage is shown to be a function of L_x/L_y^{ϕ}, with $\phi \simeq 0.2$. Thus, depending on this parameter, the resulting state could be either single or multistriped. In the second, stripe coarsening stage, the coarsening time is found to be proportional to L_y, becoming infinitely long in the thermodynamic limit. This implies that the multistriped state is thermodynamically *stable*. The results put previous studies of the model in a more general framework.

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I. INTRODUCTION

Driven diffusive systems have been extensively studied in recent years. They serve as a fruitful framework for studying the statistical mechanics of systems far from thermal equilibrium. Driven by an external field these systems reach a steady state with a non-vanishing current and as such do not satisfy detailed balance. Studies of these models have revealed many differences between such systems and systems in thermal equilibrium. For example, several one-dimensional driven diffusive systems with local dynamics exhibit long range order and spontaneous symmetry breaking. Such phenomena cannot occur in thermal equilibrium when the interactions are short ranged [1,2].

Many studies of driven diffusive systems have focused on a driven lattice-gas (Ising) model. The model was introduced by Katz, Lebowitz, and Spohn [3] and is often referred to as the ''standard model.'' In $d=2$ dimensions the model is defined on an $L_x \times L_y$ lattice. Each of the lattice sites i, is either occupied by a particle or is vacant. A macroscopic configuration is characterized by a set of occupation numbers $\{n_i\}$ where $n_i=0,1$ represents a vacant or an occupied site, respectively. Usually the model is studied with an equal number of occupied and vacant sites. An energy $\mathcal{H}=-\Sigma_{\langle ij\rangle} n_i n_j$ is associated with each configuration. Here the sum is over $\langle ij\rangle$ nearest neighbor sites. The energy represents an attractive interaction between the particles. Periodic boundary conditions are imposed in both directions. An external drive is introduced through a field E that biases the motion of the particles in the $-y$ direction. The periodic boundary conditions in this direction results in a current of particles through the system along the field direction. Specifically the dynamics of the model is defined through the exchange of nearest neighbor particles with a rate

$$W=\min\{1,\exp(-\beta\Delta\mathcal{H}-E\Delta y)\}. \tag{1}$$

Here β is an inverse temperaturelike parameter, and $\Delta y=(-1,0,1)$ for a particle attempting to hop along, orthogonal to, or against the direction of the driving field. The energy difference between the two configurations after and before the particle exchange is denoted by $\Delta\mathcal{H}$.

The model has been studied extensively for nearly two decades [4]. Monte Carlo simulations suggest that the (T,E) phase diagram of the model is composed of two phases: A high temperature disordered phase in which the particle density is homogeneous, and a low temperature phase in which the system orders and phase separates into high density and low density regimes. It was found that in this phase the particles evolve towards a striped structure parallel to the direction of the driving field. Numerical studies indicate that a slow coarsening takes place in this state [5–8]. As the magnitude of the driving field is increased, the transition temperature between the two phases increases and saturates at about $1.41T_O$ [9], where T_O is the Onsager temperature corresponding to $E=0$.

Recent Monte Carlo simulations of this model suggest that the evolution of the striped phase is rather complex. For a square system the stripes are found not to coarsen in the thermodynamic limit, yielding a multistriped ordered state. This phase was termed extraordinary or ''stringy'' [10]. On the other hand systems with large aspect ratio, $L_y \gg L_x$, were found to evolve toward a single stripe phase.

In order to get a better understanding of the nature of the ordered phase of the driven lattice gas model we carry out in this paper a finite size scaling analysis of the evolution process starting from a fully disordered state. We find that the model evolves via two stages:

(a) an early *stripe formation* stage in which stripes are formed from the initially disordered state; and

(b) a *stripe coarsening* stage in which the multistripe configuration formed in the early stage coarsen by reducing the number of stripes and increasing their average width. A typical evolution of such a system is shown in Fig. 1.

Our studies yield two main results:

(1) The number of stripes that are formed at the end of the initial stripe formation stage strongly depends on the aspect ratio of the system. In particular we find that the number of stripes m scales as $m \sim L_x/L_y^{\phi}$, with $\phi \simeq 0.2$. This implies that for narrow systems $(L_x/L_y^{\phi} \lesssim 1)$ a single stripe is formed

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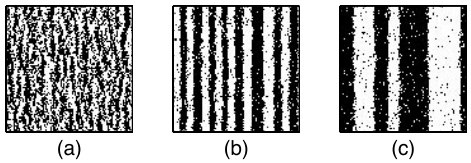

FIG. 1. A typical evolution of a system of size $L_x=L_y=100$ from a random initial condition. Configurations from times (a) 50, (b) 2000, and (c) 500 000 Monte Carlo sweeps are shown. Here $\beta=2$ and $E=\infty$. One can clearly observe the two steps of the coarsening process described in the text.

at the end of the first stage, while for wide systems ($L_x/L_y^{\phi} \gg 1$) the resulting structure is multistriped.

(2) Simple arguments are presented to show that during the stripe coarsening stage the average width of the stripes grows with time as $(t/L_y)^{1/3}$. This behavior is verified by extensive numerical simulations. Therefore, the coarsening of the stripes becomes slower as the system size in the direction of the drive is increased. This implies that in the thermodynamic limit a multiple striped configuration is in fact stable. We note that similar phenomena of arrested striped configurations have been observed in previous studies of coarsening of other models with striped structures perpendicular to the direction of the drive [11].

The paper is organized as follows: In Sec. II the stripe formation stage is discussed. Section III considers the stripe coarsening stage. We end with a summary and discussion of the implications of our results to other related works in Sec. IV.

II. THE STRIPE FORMATION STAGE

The evolution of the driven lattice gas model in the early stripe formation stage has received some attention [6–8]. Numerical simulations indicate that the domain growth process that takes place in this stage is highly anisotropic. The typical domain size in the direction of the drive and the direction perpendicular to it grow differently. In particular it has been observed [6,7] that the typical domain size parallel to the drive grows roughly as $\ell_y \sim t^{\varphi_\parallel}$ with $\varphi_\parallel \simeq 1$, while the typical domain size perpendicular to the drive grows roughly as $\ell_x \sim t^{\varphi_\perp}$, $\varphi_\perp \simeq 0.2$. This behavior is very different from that of a nondriven system evolving towards equilibrium. It is well known that in such a system, when the dynamics is conserving, as is the case here, the average linear domain size ξ grows as $t^{1/3}$ [12]. The difference in behavior is due to the inherent anisotropy induced by the drive.

The number of stripes formed in the system at the end of the stripe formation stage can be estimated using the results described above. For a stripe to form in the system the size of a domain along the direction of the drive $\ell_y(t)$ must be of the order of the system size L_y. Since $\ell_y(t) \sim t^{\varphi_\parallel}$ the time for this to occur, t_s scales, as $t_s \sim L_y^{1/\varphi_\parallel}$. At this time the typical domain size perpendicular to the drive is

$$\ell_x(t_s) \sim t_s^{\varphi_\perp} \sim L_y^{\varphi_\perp/\varphi_\parallel}. \tag{2}$$

Thus, the number of stripes formed, m, scales as

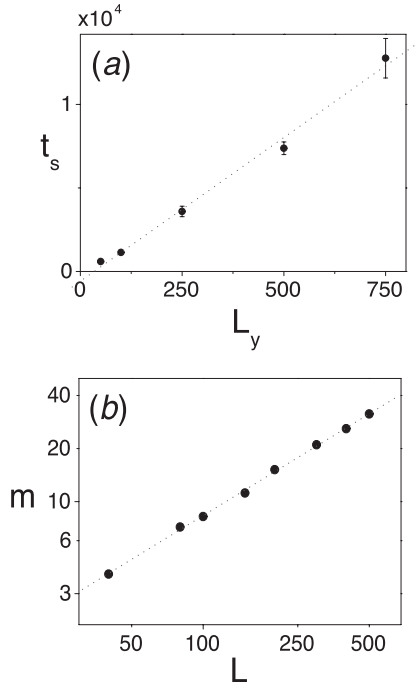

FIG. 2. (a) The stripe formation time t_s (in Monte Carlo sweeps) plotted against L_y, for systems with $L_x=100$. The behavior is consistent with $\varphi_\parallel \simeq 1$. (b) The number of stripes formed at the end of the first stage in square systems of various sizes $L \equiv L_x = L_y$, plotted on a log-log scale. The straight line corresponds to $m \sim L^{0.82}$. Here the error estimate for each measurement is of order 1.

$$m \sim \frac{L_x}{\ell_x(t_s)} \sim \frac{L_x}{L_y^{\phi}}, \tag{3}$$

where $\phi = \varphi_\perp/\varphi_\parallel$. Using the estimates for the exponents $\varphi_\perp$ and $\varphi_\parallel$ one has $\phi \simeq 0.2$.

Specifically, for a square system, where $L_x=L_y \equiv L$, Eq. 3 implies $m \sim L^{1-\phi}$. Since $\phi<1$ we find that the number of stripes grows as the system size is increased, and one always reaches a multistriped state. The stripe density, m/L_x, vanishes in the thermodynamic limit.

To verify these results Monte Carlo simulations are performed for various system sizes, starting from a random initial condition. The Monte Carlo procedure we use is standard. At each time step a pair of neighboring sites is chosen randomly and updated according to the rate W given in Eq. (1). Throughout the paper we use $E=\infty$ and $\beta=2$, for which the system is ordered. We have checked that the main features of this study are unchanged for other values of the parameters as long as the system is in the ordered phase. We first verify the growth law of t_s with L_y. In order to evaluate t_s, the equal-time correlation of two sites at a distance $L_y/2$ in the drive direction is measured and averaged over the sample. The time t_s is estimated by the time at which the average measured correlation reaches the value of 0.4. The results, averaged over about 100 samples for each system size, are shown in Fig. 2(a). One can see that the behavior of t_s with L_y is consistent with $\varphi_\parallel \simeq 1$.

The number of stripes initially formed in the system is estimated by performing a Fourier transform of the density in the x direction and locating its first peak at a nonzero wave-

length. This procedure is repeated 40 times for each system size. For simplicity we consider only square systems. In Fig. 2(b) we plot the location of the peak for square systems as a function of L. The fitted exponent for slightly over a decade of system sizes gives $\phi \approx 0.18 \pm 0.05$ that fits rather well with the values predicted by the argument $\phi \approx 0.2$. The linear dependence of m on L_x for nonsquare systems is also verified through simulations that are not shown in this paper.

In general, the number of stripes also depends on the magnitude of the driving field and the temperature. One can write, based on Eq. 3, $m = AL_x/L_y^{\phi}$, where the amplitude $A(E,\beta)$ is introduced. We find that the amplitude is an increasing function of both the magnitude of the driving field E and the inverse temperature β. We note that when $L_y^{\phi} \geqslant AL_x$, the number of stripes already at the end of the first stage is expected to be one.

III. THE STRIPE COARSENING STAGE

We now turn to the second stage of the ordering process, namely, the coarsening of the stripes formed in the early stage. We present a simple argument suggesting that the time t in which stripes coarsen scales linearly with the system size parallel to the drive, L_y. Moreover, we show that the average stripe width $\ell(t)$ scales with time as $(t/L_y)^{1/3}$. The fact that the characteristic time associated with the coarsening process scales linearly with L_y implies that in the thermodynamic limit, where $L_y \to \infty$, the coarsening time becomes infinite and thus the multistripe structure exists as a thermodynamically stable state. Our argument relies on two main features of the driven system: (*i*) the fact that the ordered domains, namely the stripes, are of the size of the system, and (*ii*) the smoothness of the domain walls bounding the stripes. This last feature has been shown to be a result of the drive [13–16]. In contrast to the nondriven two-dimensional Ising model, where the domain walls may be rough, here the driving field makes the domain walls smooth.

We proceed by considering a striped state composed of alternating stripes of particles and vacancies with average width ℓ. Neighboring stripes of particles interact with each other by an exchange of particles. Since the boundaries of the stripes are smooth, the lateral distance that particles have to travel in order to move from one stripe to the other is of the order of ℓ. To estimate the coarsening time we assume that within a stripe of vacancies the density of particles is low enough so that the particles may be considered as noninteracting. This assumption is qualitatively supported by the configurations observed in simulations (see, e.g, Fig. 1). When a particle reaches the boundary it is absorbed in the neighboring particle stripe. Thus the lateral motion of the particles within a stripe of vacancies can be considered as a one-dimensional random walk in the x direction with two absorbing walls located at $x=0$ and $x=\ell$. This problem is known as the gambler's ruin problem [17]. The probability of such a particle to move from 0 to ℓ is given by $p(\ell) \sim 1/\ell$.

For the width of a stripe to decrease by one lattice spacing, it has to lose L_y particles. Due to the right-left symmetry of the problem, the particle currents from one stripe to the other are balanced on average. Therefore a net transfer of particles from one stripe to another is only due to fluctuations in the lateral current. The net excess in the number of particles transferred at a time interval t is then proportional to $\sqrt{L_y p(\ell) t}$. For one stripe to shrink and disappear ℓL_y particles must be transferred so that

$$\sqrt{L_y p(\ell) t} \sim \ell L_y \quad . \tag{4}$$

Combining this result with $p(\ell) \sim 1/\ell$ one finds that the average stripe width in the system grows as

$$\ell(t) \sim \left(\frac{t}{L_y} \right)^{1/3} . \tag{5}$$

This suggests that the coarsening time scales with L_y, yielding a stable striped structure in the thermodynamic limit.

The scaling form (5) may be verified numerically by studying the two point particle-particle correlation function. To carry out this analysis we note that in an isotropic system without a driving field, the coarsening process is characterized by a single length scale $\xi(t)$, which could be the linear size of the growing domains. In this case the two-point particle-particle correlation function obeys a scaling form [12]

$$C(r,t) = g\left(\frac{r}{\xi(t)} \right), \tag{6}$$

where r is the distance between two points. Driven systems, on the other hand, are non-isotropic, and correlations along the drive and perpendicular to it behave differently. The typical length scale perpendicular to the drive is given by Eq. (5). Thus we expect the correlation function in the x direction to be of the form

$$C_{\perp}(x,t) = g_{\perp}\left(\frac{x}{(t/L_y)^{1/3}} \right). \tag{7}$$

The asymptotic behavior of $g_{\perp}(z)$ for $z \to 0$ is expected to obey Porod's law, which states that $g_{\perp}(z) = 1/2 - \eta z$ with some constant η. For $z \to \infty$ one should have $g_{\perp}(z) \to 1/4$.

We now turn to describe numerical studies that support our results. Note that the scaling variable in Eq. (7) involves three parameters. All three parameters are varied in our numerical studies. This is a demanding computational task, and a good collapse of the data is a strong conformation of the scaling analysis. In these studies an initial striped configuration along the drive direction y is considered, and its evolution is simulated. The two point correlation function $C_{\perp}(x,t)$ is then calculated, and is shown to obey the scaling form (7). The widths of the stripes in the initial configuration are randomly chosen from a Poisson distribution with a mean width ℓ_0. The simulations are performed for lattices of three different sizes: 960×8, 800×16, and 960×32. We consider several values of L_x to demonstrate that this parameter does not play an important role in the process. The mean width of the stripes in the initial configurations is taken to be $\ell_0 = 4$. The two-point particle-particle correlation function $C_{\perp}(x,t)$

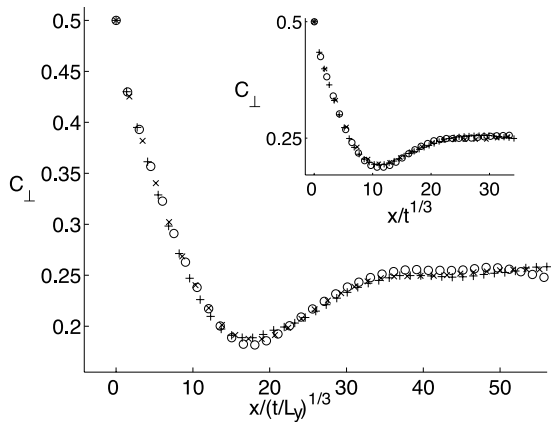

FIG. 3. The two-point particle-particle correlation function $C_\perp$ is plotted as a function of the scaling variable $x/(t/L_y)^{1/3}$, for systems of size 960×8 (marked by $\times$), 800×16 ($\bigcirc$), 960×32 ($+$). The times of measurement are chosen arbitrarily ($t=1,3,8\times10^6$ Monte Carlo sweeps, respectively). In the inset, $C_\perp$ as a function of $x/t^{1/3}$ is shown for a system of size 960×8 and times $t=0.2(\times)$, $1(\bigcirc)$, $2(+)\times10^6$ Monte Carlo sweeps.

is measured and averaged over 110, 75, and 54 simulations for $L_y=8$, 16, and 32, respectively.

The scaling form (7) suggests that data collapse should take place with respect to the three variables x, t, and L_y. This collapse is checked in two steps. First we consider $L_y=8$ and show that $C_\perp$ is a function of $x/t^{1/3}$ as expected. This is demonstrated in the inset of Fig. 3. Similar results are obtained for the other system sizes as well.

Next, we verify the full scaling form (7). In Fig. 3 correlation functions for the three different system sizes are plotted. For each system size the correlation function is evaluated for arbitrarily chosen t and the data is then plotted as a function of the scaling variable $x/(t/L_y)^{1/3}$. Again, the quality of the data collapse supports our main result. Although computation time limits us to a relatively small systems, we believe the quality of the data backs our scaling argument.

IV. DISCUSSION

The evolution of the driven lattice-gas model was considered starting from a random initial configuration. We have shown, using simple scaling arguments and extensive numerical simulations, that the evolution proceeds via two stages: an early, stripe formation stage in which stripes of the size of the system are formed, followed by a second stage in which the stripes coarsen. While the first stage lasts $t_s\sim L_y$, the system evolves towards a single stripe configuration in the second stage at a time of order $\sim L_x^3 L_y$. This is a result of the fact that the typical width of stripes in the coarsening stage scales with time as $\ell(t)\sim(t/L_y)^{1/3}$. This result indicates that the coarsening time of multistriped configurations scales with the system length L_y, suggesting that these configurations exist as stable states in the thermodynamic limit $L_y\to\infty$.

Thus, starting from a random initial configuration, the system evolves to one of two types of states, depending on its aspect ratio. For $L_x/L_y^{\phi}\leqslant1$ ($\phi\simeq0.2$) the stripe formation stage leads directly to a single stripe state, while for $L_x/L_y^{\phi}\gg1$ multistriped states are reached. The coarsening process of these states proceeds with a time scale proportional to L_y.

These results put in a more general framework previous studies of this model that considered either the early stages [6–8] of the evolution or the nature of the steady state [4]. A recent study of a square system has shown [10] that a multistriped state (termed ''stringy'') is reached from a random initial condition. It was suggested that this state is stable. Our studies indicate that this is indeed the case for an infinitely large system. However we expect a finite system to coarsen to a single striped state at a time of the order of $L_x^3 L_y$. The fact that the steady state of a system with a small aspect ratio, L_x/L_y, was found to be composed of a single stripe is consistent with our scaling picture.

Finally, note that the slow coarsening of the stripes is a direct consequence of stripes spanning the entire system. This is a result of the existence of the drive, and is expected to be valid also in higher dimensions. It would be interesting to study such processes in high-dimensional systems. We note, however, that already in two dimensions the computational effort was considerable.

ACKNOWLEDGMENTS

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